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The Dirac equation and integrable systems of KP type

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Abstract. The propagator for the 2D heat equation in an arbitrary linear space is shown to give solutions of the two-component Kadomtsev–Petviashvilii (KP) equations, also called Davey–Stewartson system. This propagator is subject to the Klein–Gordon equation and its right-derivatives are required to be of rank one, that imply that it can be expressed in terms of solutions of the Dirac equation. Large families of solutions of the two-component Kadomtsev–Petviashvilii equations are constructed in terms of solutions of the heat and Dirac equations. Particular attention is paid to the real reductions of the Davey–Stewartson type, recovering in this way the line solitons and the multidromion solutions. Moreover, new solutions to the Davey–Stewartson I are presented as massive deformations of the dromion.

1. Introduction

It is the aim of the present work to show that the free propagator for the heat equation in two dimensions, with some conditions, determines a solution of the nonlinear two-component Kadomtsev–Petviashvilii (KP) and related equations. Observe that was in [15] where the two-component KP equation was introduced and it was also noted that it reduces to a two-dimensional nonlinear Schrödinger equation [21], the well known Davey–Stewartson (DS) equation [4]. This motivates the alternative name of DS system given in [16] to the two-component KP. Nevertheless, for historical reasons we retain the terminology introduced in [15].

Previous results in this direction [12] derived from the following construction. Let $\psi(x, y, t)$ be an invertible transformation on a linear space V depending on the three complex variables x, y, t. Assume ψ satisfies the linear equations

$$\psi_y = \psi_{xx} \qquad \psi_t = \psi_{xxx}$$

then the linear transformation on V defined as $F(x, y, t) = \psi_x(x, y, t) \cdot \psi(x, y, t)^{-1}$ solves the potential KP equation for F, $(F_t - \frac{1}{4}F_{xxx} - \frac{3}{2}F_x^2)_x = \frac{3}{4}F_{yy} - \frac{3}{2}[F_x, F_y]$. If one wants to have a scalar potential KP equation it is enough to impose the additional restriction on F, $F = A + e \otimes \alpha$, where $A \in \mathrm{L}(V)$, $e \in V$ are constant elements and $\alpha(x, y, t) \in V^*$. That means F be a rank-one operator modulo a constant operator. In that

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case the function $f(x,y,t) = \langle\alpha(x,y,t),e\rangle$ is a solution of the potential KP equation, $(f_t - \frac{1}{4}f_{xxx} - \frac{3}{2}f_x^2)_x = \frac{3}{4}f_{yy}$. The new condition reflects on ψ through the relation $\psi_x \cdot \psi^{-1} = A + e \otimes \alpha$ or according to $\psi_x = A\psi + e \otimes \beta$ once we define $\beta = \alpha\psi$. For this linear functional we find the equations $\beta_y = \beta_{xx}$, $\beta_t = \beta_{xxx}$ which parametrize the space of solutions ψ for which $f = \langle\beta, \psi^{-1}e\rangle$ gives a solution of the potential KP equation.

Letting ψ depend on more spatial variables $x_1, x_2, \ldots$ (here we shall be interested in two variables x_1, x_2) would result in multicomponent KP equations. One proceeds along the same lines outlined above, but now with two coordinates x_1, x_2, vectors e_1, e_2 and two constant operators A_1, A_2, to get the two-component KP equation from the flow defined by the heat equation $\psi_t = \Delta\psi$ (the time t occupies the place of the variable y in the conventional KP). In [13] there were obtained corresponding formulae for the two scalar functions p_{12}, p_{21}, instead the single f previously considered, that are of the form $p_{12} = \langle\beta_1, \psi^{-1}e_2\rangle$, $p_{21} = \langle\beta_2, \psi^{-}e_1\rangle$, where we denote by $\beta_1(x_1,t)$ and $\beta_2(x_2,t)$ two solutions of the heat equation in V^*. That construction lay on the algebraic relations $A_1A_2 = A_2A_1 = 0$, $A_1e_2 = A_2e_1 = 0$ which in turn gave heat equations for the vectors $\beta_1(x_1,t), \beta_2(x_2,t)$ which make up the solutions p_{12} and p_{21}. In fact the propagator $\psi(x_1,x_2,t) \in \mathrm{GL}(V)$ was of the form $\psi_1(x_1,t) + \psi(x_2,t)$. For related formulae for the solutions obtained by these methods see [6, 7, 9, 10, 11, 17, 18, 19, 20, 22].

This paper deals with the problem of determining under which conditions the free propagator of the heat equation would continue to give solutions of the two-component KP. The answer is that one can replace the algebraic relations written above by the less restrictive equations $A_1A_2 = A_2A_1$, $A_1e_2 = \lambda_{12}e_2$, $A_2e_1 = \lambda_{21}e_2$ to keep the p_{ij} as solutions. Now, however, one finds that the formulae expressing the new p_{ij} differs by the additive constants λ_{ij} from the previous ones and that the equations satisfied by the β_i's is the heat equation in two dimensions. Thus β_1 and β_2 now depend on the two variables x_1 and x_2 and are connected by the Dirac equation with mass. When this mass goes to zero one recovers the preceding situation of separated variables in the propagator ψ and the vectors β_i. A remarkable aspect of the new setting that deserves to be mentioned is the description of the solutions p_{ij} by elementary equations of mathematical physics. These are the heat and Klein–Gordon equation for the propagator ψ and the Dirac equation for the vectors β_i. The appearance of mass has the effect of modifying the asymptotic behaviour of the p_{ij} as it appears when examining dromion solutions of the DS equation, a real reduction of the two-component KP.

Besides the system considered until now, one can consider its reduction to a single function that satisfies the DS equation we alluded to before. This amounts to the replacement by real variables of the complex variables x_1, x_2 and the substitution of the heat equation by a Schrödinger equation. In that case we find that for the concrete solutions there are linear constraints resulting in the DSI and DSII variants of the Davey–Stewartson equation.

The organization and contents of the paper are as follows. In section 2 we derive the two-component KP equation from the heat equation and appropriate linear constraints that can be described by means of the Klein–Gordon equation and the Dirac equations. In this context we obtain a Lax pair formulation. The construction of solutions is treated in section 3 where we analyse some examples in the simplest cases. It ends with a treatment of the reduction to the nonlinear Schrödinger (NLS) system. Finally, we study in section 4 the reduction problem and give sufficient conditions for the construction of solutions of the DS equation. For the DSI reduction the line soliton as well as the multidromion solution are recovered. Moreover we give new solutions, which we have called massive deformations of dromions.

2. The two-component KP equations

In this section we shall study automorphisms $\psi(x_1, x_2, t) \in \mathrm{GL}(V)$ of some complex linear space V that depend on the complex variables x_1, x_2, t according to the heat equation

$$(\partial_t - \Delta)\psi = 0, \tag{2.1}$$

where $\Delta := \partial_1^2 + \partial_2^2$ is the Laplacian and $\partial_i = \partial/\partial x_i$. Then we can think of ψ as a representation of the free propagator of the heat equation in the space V, the flow associated with the Laplacian Δ.

Besides equation (2.1), which takes place in the Lie group $\mathrm{GL}(V)$, we are interested in the flows induced on the Lie algebra $\mathrm{L}(V)$ of linear operators on V. To this end we consider the right-derivatives

$$F_i := \partial_i\psi \cdot \psi^{-1} \tag{2.2}$$

for which the zero-curvature condition they satisfy (which holds by construction) can be written as

$$\partial_2 F_1 + F_1 F_2 = \partial_1 F_2 + F_2 F_1 =: Q, \tag{2.3}$$

where we have also defined the operator function $Q(x_1, x_2, t) \in \mathrm{L}(V)$. Observe that Q admits also the expression $Q = \partial_1\partial_2\psi \cdot \psi^{-1}$, or ψ and Q are connected by a generalized Klein–Gordon equation

$$(\partial_1\partial_2 - Q)\psi = 0.$$

It is easy to see that an automorphism $\psi(x_1, x_2, t) \in \mathrm{GL}(V)$ satisfies (2.1) if and only if its right-derivatives $F_i(x_1, x_2, t)$ as defined in (2.2) are solutions of

$$\partial_t F_1 = (\partial_1^2 - \partial_2^2)F_1 + 2(\partial_1 F_1 \cdot F_1 - F_1 \cdot \partial_2 F_2) + 2\partial_2 Q$$

$$\partial_t F_2 = -(\partial_1^2 - \partial_2^2)F_2 + 2(\partial_2 F_2 \cdot F_2 - F_2 \cdot \partial_1 F_1) + 2\partial_1 Q.$$

The case of interest in connection with integrable systems appears when the operator function Q is a constant operator, a condition that will hold from now on.

Proposition 1. The two conditions for the automorphism ψ

$$(\partial_t - \Delta)\psi = 0 \tag{2.4}$$

$$(\partial_1\partial_2 - Q)\psi = 0 \tag{2.5}$$

where $Q \in \mathrm{L}(V)$ is a constant operator, are equivalent to the following equations for the right-derivatives F_1, F_2 of ψ:

$$\partial_2 F_1 + F_1 F_2 = Q \tag{2.6}$$

$$\partial_1 F_2 + F_2 F_1 = Q \tag{2.7}$$

$$\partial_t F_1 = (\partial_1^2 - \partial_2^2)F_1 + 2(\partial_1 F_1 \cdot F_1 - F_1 \cdot \partial_2 F_2), \tag{2.8}$$

$$\partial_t F_2 = -(\partial_1^2 - \partial_2^2)F_2 + 2(\partial_2 F_2 \cdot F_2 - F_2 \cdot \partial_1 F_1). \tag{2.9}$$

Equations (2.6)–(2.9), as we shall show, can be considered as an operator extension of the two-component KP equations. In fact the two-component KP equations arise if we impose rank-one constraints on the right-derivatives F_i of ψ.

Definition 1. The rank-one constraints are defined by the equations:

$$\partial_i\psi(x_1,x_2,t) = (A_i + e_i\otimes\alpha_i(x_1,x_2,t))\psi(x_1,x_2,t) \qquad i=1,2 \tag{2.10}$$

where $A_i\in \mathrm{L}(V)$ are constant operators subject to

$$A_1A_2 = A_2A_1 = Q \tag{2.11}$$

with $Q\in\mathrm{L}(V)$ a constant operator, $e_i\in V$ are constant vectors such that

$$A_1e_2 = \lambda_{12}e_1 \qquad A_2e_1 = \lambda_{21}e_2 \qquad \lambda_{12},\lambda_{21}\in\mathbb{C} \tag{2.12}$$

and $\alpha_i(x_1,x_2,t)\in V^*$ are linear functionals depending on x_1,x_2,t.

We now introduce the functions in terms of which the two-component KP equations are defined.

Definition 2. The functions p_{ij} are given by

$$p_{ij} := \lambda_{ij} + \langle\alpha_i, e_j\rangle \qquad i\neq j \tag{2.13}$$

and

$$p_{11} := -\lambda_{12}\lambda_{21}x_2 + \langle\alpha_1, e_1\rangle \tag{2.14}$$

$$p_{22} := -\lambda_{12}\lambda_{21}x_1 + \langle\alpha_2, e_2\rangle. \tag{2.15}$$

Then equations (2.3) and (2.10) imply

$$\begin{aligned}\partial_2\alpha_1 + p_{12}\alpha_2 + \alpha_1A_2 &= 0\\ \partial_1\alpha_2 + p_{21}\alpha_1 + \alpha_2A_1 &= 0.\end{aligned} \tag{2.16}$$

Upon contraction of (2.6) and (2.7) with e_1 and e_2 we deduce that if ψ is a solution of (2.5) and (2.10), then the functions p_{ij} given in definition 2 are solutions of the equations

$$\partial_2p_{11} + p_{12}p_{21} = 0 \tag{2.17}$$

$$\partial_1p_{22} + p_{12}p_{21} = 0 \tag{2.18}$$

that imply the existence of a local potential, say U, in terms of which

$$p_{ii} = \partial_iU$$

fulfilling the relation

$$p_{12}p_{21} + \partial_1\partial_2U = 0.$$

The evolution for the α_i is determined by (2.10) for the right-derivatives together with (2.8), (2.9) and (2.6), (2.7) that give

$$\begin{aligned}\partial_t\alpha_1 &= (\partial_1^2 - \partial_2^2)\alpha_1 + 2\partial_1\alpha_1\,A_1 + 2\partial_1p_{11}\,\alpha_1 - 2p_{12}\partial_2\alpha_2\\ \partial_t\alpha_2 &= -(\partial_1^2 - \partial_2^2)\alpha_2 + 2\partial_2\alpha_2\,A_2 + 2\partial_2p_{22}\,\alpha_2 - 2p_{21}\partial_1\alpha_1.\end{aligned} \tag{2.19}$$

Contracting them with e_1 and e_2 one finds the following:

Proposition 2. If $\psi(x_1, x_2, t)$ is a solution of (2.5), (2.4) with right-derivatives as in (2.10) then p_{12}, p_{21} as given in definition 2 and the potential U satisfy the system of equations

$$\partial_1\partial_2 U + p_{12}p_{21} = 0 \tag{2.20}$$

$$\partial_t p_{12} - (\partial_1^2 - \partial_2^2)p_{12} - 2p_{12}(\partial_1^2 - \partial_2^2)U = 0 \tag{2.21}$$

$$\partial_t p_{21} + (\partial_1^2 - \partial_2^2)p_{21} + 2p_{21}(\partial_1^2 - \partial_2^2)U = 0\,. \tag{2.22}$$

This system is that of the well known two-component KP equations.

Therefore as a first result we have obtained a description of them by means of the *free* equations (2.4) and (2.5) for a ψ subject to the linear constraints (2.10). The functions p_{ij} satisfy the equations defining the Darboux coefficients or rotation coefficients for an orthogonal system of curvilinear complex coordinates on a complex surface. The time evolution, as given by the two-component KP equations, represents an integrable deformation of them [16].

Let us recall that the Lax equations [16] are one of the following linear systems ($\tilde{h}$ denoting the adjoint of h) for the wavefunctions $h = (h_1, h_2)^t$ and $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)^t$:

$$\begin{pmatrix} \partial_2 & p_{12} \\ p_{21} & \partial_1 \end{pmatrix} h = 0 \qquad \begin{pmatrix} \partial_2 & p_{21} \\ p_{12} & \partial_1 \end{pmatrix} \tilde{h} = 0 \tag{2.23}$$

$$\begin{pmatrix} \partial_t - \partial_1^2 + \partial_2^2 - 2\partial_1^2 U & 2p_{12}\partial_2 \\ 2p_{21}\partial_1 & \partial_t + \partial_1^2 - \partial_2^2 - 2\partial_2^2 U \end{pmatrix} h = 0$$
$$\begin{pmatrix} \partial_t + \partial_1^2 - \partial_2^2 + 2\partial_1^2 U & -2p_{21}\partial_2 \\ -2p_{12}\partial_1 & \partial_t - \partial_1^2 + \partial_2^2 + 2\partial_2^2 U \end{pmatrix} \tilde{h} = 0 \tag{2.24}$$

and that the two-component KP equations are a consequence of the compatibility conditions on h or $\tilde{h}$. Here the coefficients h_i give a diagonal metric of zero-curvature $h_1^2 \mathrm{d}x_1^2 + h_2^2 \mathrm{d}x_2^2$, and their t-dependence provides an integrable deformation of that metric.

We have two different methods for constructing wavefunctions, depending on whether we consider the vector space V (which uses the dressed propagator ψ and gives $\tilde{h}$), or its dual V^* for the second method giving h, which relies on the functionals α_1 and α_2 and also on the vacuum propagator ψ_0 constructed in terms of the operators A_1 and A_2 as

$$\psi_0(x_1, x_2, t) := \exp[A_1 x_1 + A_2 x_2 + (A_1^2 + A_2^2)t] \tag{2.25}$$

(i) The first method identifies $\tilde{h}_i$ with the vector functions taking values in V

$$\tilde{h}_i := \psi^{-1} e_i \tag{2.26}$$

that satisfy the (2.23) and (2.24), as follows from the relations

$$\partial_i(\psi^{-1}) + \psi^{-1}(A_i + e_i \otimes \alpha_i) = 0$$

$$\partial_t(\psi^{-1}) + \left(\sum_{i=1,2} (\psi^{-1} e_i \otimes \partial_i \alpha_i - \partial_i \psi^{-1}(A_i + e_i \otimes \alpha_i)\right) = 0$$

when applied to the vectors e_1, e_2.

(ii) The linear functionals

$$h_i := \alpha_i \psi_0 \tag{2.27}$$

also satisfy equations (2.23) and (2.24), as one can deduce from equations (2.16).

3. Construction of solutions

As we have seen, the two-component KP equations can be solved in terms of a function $\psi(x_1, x_2, t) \in \mathrm{GL}(V)$. In order to solve the equations defining ψ in proposition 1 together with the constraints (2.10), we define the new linear functionals $\beta_i := \alpha_i \psi$.

We shall show that such a $\beta \in \mathbb{C}^2 \otimes V^*$, with components β_i, $\beta := (\beta_1, \beta_2)^t$, solves the Dirac equation in a two-dimensional space-time of coordinates x_1, x_2 and *metric* given by

$$\mathrm{d}s^2 = \tfrac{1}{2}\big(\mathrm{d}x_1 \otimes \mathrm{d}x_2 + \mathrm{d}x_1 \otimes \mathrm{d}x_2\big). \tag{3.1}$$

In this case the Dirac matrices γ_1, γ_2 satisfy

$$\gamma_i^2 = 0 \quad i = 1, 2 \qquad \gamma_1\gamma_2 + \gamma_2\gamma_1 = 1$$

and an action of this Clifford algebra on $\mathbb{C}^2$ is given by

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We now introduce two masses, each which can be considered as the dual of the other, as the following bilinear expressions in the gamma matrices:

$$m_\gamma := \mathrm{diag}(\lambda_{12}, \lambda_{21}) = \sum_{i,j} \gamma_i\gamma_j\lambda_{ij}$$

$$\tilde{m}_\gamma := \mathrm{diag}(\lambda_{21}, \lambda_{12}) = \sum_{i,j} \gamma_i\gamma_j\lambda_{ji}$$

and the corresponding Dirac operator is given by $\partial_\gamma := \gamma_1\partial_1 + \gamma_2\partial_2$.

Note that equation (2.10) can be written in the form

$$\partial_i\psi = A_i\psi + e_i \otimes \beta_i \tag{3.2}$$

so that

$$\partial_i\partial_j\psi = A_i\partial_j\psi + e_i \otimes \partial_j\beta_i.$$

In particular, when $i \neq j$ the above equation reads

$$Q\psi = A_1\partial_2\psi + e_1 \otimes \partial_2\beta_1$$

$$Q\psi = A_2\partial_1\psi + e_2 \otimes \partial_1\beta_2$$

which, bearing in mind definition 1, results in the Dirac equation for the spinor coefficients β_i:

$$\partial_1\beta_2 + \lambda_{21}\beta_1 = 0$$

$$\partial_2\beta_1 + \lambda_{12}\beta_2 = 0$$

equations that can be condensed to

$$(\partial_\gamma + \tilde{m}_\gamma)\beta = 0.$$

The deformation equation for β follows from the ∂_t-derivative of (3.2) and the evolution (2.4) of ψ from which we deduce the relation

$$(\partial_i - A_i)\Delta\psi = e_i \otimes \partial_t\beta_i$$

that gives $(\partial_t - \Delta)\beta = 0$ once we replace $(\partial_i - A_i)\psi$ by $e_i \otimes \beta_i$ according to (2.10).

Observe in particular that β is a solution of the Klein–Gordon equation

$$(\partial_1\partial_2 - \lambda_{12}\lambda_{21})\beta = 0.$$

In order to construct ψ we select the vector functions

$$b_i := \psi_0^{-1} e_i$$

with ψ_0 given in (2.25). The vector function $b := (b_1, b_2)^t$ with values in $\mathbb{C}^2 \otimes V$ satisfies

$$(\partial_\gamma + m_\gamma) b = 0 \tag{3.3}$$

$$(\partial_t + \Delta) b = 0 \tag{3.4}$$

but these two equations do not characterize the functions $b_i = \psi_0^{-1} e_i$ completely. We are now in a position to describe ψ in terms of β.

Proposition 3. Let $\beta_i(x_1, x_2, t) \in V^*$, $i = 1, 2$, be functions of x_1, x_2, t taking values in V^*, such that $\beta = (\beta_1, \beta_2)^t$ is a solution of

$$(\partial_\gamma + \tilde{m}_\gamma)\beta = 0$$

$$(\partial_t - \Delta)\beta = 0.$$

A solution $\psi(x_1, x_2, t)$ to (2.5), (2.4) and (2.10) can be represented as

$$\psi = \psi_0 \cdot \varphi$$

where $\varphi(x_1, x_2, t)$ is determined by the system of compatible equations

$$\partial_i \varphi = b_i \otimes \beta_i \qquad i = 1, 2 \tag{3.5}$$

$$\partial_t \varphi = \sum_{i=1,2} \left[b_i \otimes \partial_i \beta_i - \partial_i b_i \otimes \beta_i \right]. \tag{3.6}$$

The α's are defined by the relations $\alpha_i = \beta_i \psi$.

Proof. Just rewrite the equations for ψ in terms of β. □

Thus, given an initial point (x_1^0, x_2^0, t_0), where φ takes the value φ_0, and an appropriate path $\gamma : [s_0, s_1] \to \mathbb{C}^3$, $\gamma(s) = (x_1(s), x_2(s), t(s))$, connecting this point with (x_1, x_2, t), the function φ can be expressed as

$$\varphi(x_1, x_2, t) = \varphi_0 + \int_{s_0}^{s_1} \mathrm{d}s \Bigg[\frac{\mathrm{d}x_1}{\mathrm{d}s}(s)(b_1 \otimes \beta_1)(\gamma(s)) + \frac{\mathrm{d}x_2}{\mathrm{d}s}(s)(b_2 \otimes \beta_2)(\gamma(s)) + \frac{\mathrm{d}t}{\mathrm{d}s}(s) \sum_{i=1,2} \big(b_i \otimes \partial_i \beta_i - \partial_i b_i \otimes \beta_i \big)(\gamma(s)) \Bigg]. \tag{3.7}$$

We can choose the path γ from (x_1^0, x_2^0, t_0) to (x_1, x_2, t) by first connecting (x_1^0, x_2^0, t_0) to (x_1^0, x_2^0, t) by a straight line, and then from (x_1^0, x_2^0, t) to (x_1, x_2, t) by any path keeping t unchanged. If b_i and β_i, $i = 1, 2$ and their derivatives vanish at (x_1^0, x_2^0) for all t, then equation (3.7) simplifies to

$$\varphi(x_1, x_2, t) = \varphi_0 + \int_{s_0}^{s_1} \mathrm{d}s \left[\frac{\mathrm{d}x_1}{\mathrm{d}s}(s)(b_1 \otimes \beta_1)(\gamma(s), t) + \frac{\mathrm{d}x_2}{\mathrm{d}s}(s)(b_2 \otimes \beta_2)(\gamma(s), t) \right]$$

a path integral in the plane (x_1, x_2) with $\gamma(s) = (x_1(s), x_2(s))$. In that case, the path integral can be expressed in terms of the primitives $\partial_i^{-1} = \int_{x_i^0}^{x_i} \mathrm{d}x_i$ that commute with ∂_i. In this manner one recovers the approach adopted in [13], where φ was written as

$$\varphi = C + \partial_1^{-1}(b_1 \otimes \beta_1) + \partial_2^{-1}(b_2 \otimes \beta_2)$$

for the $\lambda_{ij} = 0$ case.

We define the operators

$$\xi_{ij} := \psi + e_j \otimes \beta_i - e_j \otimes \eta_j \psi$$

$$\zeta_{ij} := \varphi + b_j \otimes \beta_i - b_j \otimes \delta_j \varphi$$

where the $\eta_j, \delta_j(x_1, x_2, t) \in V^*$ are linear functionals such that

$$\langle \eta_j, e_j \rangle = \langle \delta_j, b_j \rangle = 1 \qquad j = 1, 2.$$

Note that for a given η_j we can take $\delta_j = \eta_j \psi_0$ and in that case we have the relations $\xi_{ij} = \psi_0 \zeta_{ij}$.

We can now state the main result of this paper. The proof of it follows the ideas of [13, theorem 2].

Theorem 1. If φ, as given by (3.5) and (3.6), has a determinant then the functions

$$U = -\lambda_{12}\lambda_{21}x_1x_2 + \ln \det \varphi$$

$$p_{12} = \lambda_{12} + \langle \beta_1, \varphi^{-1} b_2 \rangle = \lambda_{12} + \frac{\det \xi_{12}}{\det \psi} = \lambda_{12} + \frac{\det \zeta_{12}}{\det \varphi}$$

$$p_{21} = \lambda_{21} + \langle \beta_2, \varphi^{-1} b_1 \rangle = \lambda_{21} + \frac{\det \xi_{21}}{\det \psi} = \lambda_{21} + \frac{\det \zeta_{21}}{\det \varphi}$$

represent a solution of (2.20)–(2.22). Moreover, the functions

$$\tilde{h}_i = \varphi^{-1} b_i$$

$$h_i = \beta_i \varphi^{-1}$$

satisfy equations (2.23) and (2.24), and their components determine the associated wavefunctions.

Note that with the characterization given for p_{12} and p_{21} the computation of the inverse of φ is avoided by the use of determinants. This is an advantage of the determinant-type expressions.

Clearly, the two-component KP equations are invariant under global phase shifts:

$$p_{12} \to \exp(\mathrm{i}\theta)\, p_{12}$$

$$p_{21} \to \exp(-\mathrm{i}\theta)\, p_{21}$$

$$U \to U$$

with $\theta \in [0, 2\pi)$. To lift this action up to the linear data we define

$$\beta_1 \to \exp(\mathrm{i}\theta/2)\beta_1$$

$$\beta_2 \to \exp(-\mathrm{i}\theta/2)\beta_2$$

$$e_1 \to \exp(-\mathrm{i}\theta/2)e_1$$

$$e_2 \to \exp(\mathrm{i}\theta/2)e_2$$

which preserve the Dirac equation for the β's and the b's if and only if the λ's transform according to

$$\lambda_{12} \to \exp(\mathrm{i}\theta)\lambda_{12}$$

$$\lambda_{21} \to \exp(-\mathrm{i}\theta)\lambda_{21}.$$

The equations describing the time evolution of β_i and b_i are obviously preserved. In this transformation A_1 and A_2 remain invariant, and as a consequence of (3.5) and (3.6) the automorphism φ do also remains invariant. One interesting consequence of this invariance is that we can choose the phase of one of the λ's arbitrarily, for example we may choose $\lambda_{12} \in \mathbb{R}$ without loss of generality.

As an example we consider $V = \mathbb{C}$, suppose that $e_1, e_2 \in \mathbb{C}$ are non-zero so that we can introduce the parameter $k = e_1/e_2$, in terms of which the operators A_1, A_2, which now are complex numbers, can be written as $A_1 = \lambda_{12}k$ and $A_2 = \lambda_{21}k^{-1}$. Then we have

$$b(x_1, x_2, t) = e \exp\big[-(\lambda_{12}kx_1 + \lambda_{21}k^{-1}x_2 + (\lambda_{12}^2k^2 + \lambda_{21}^2k^{-2})t)\big]$$

$$\beta(x_1, x_2, t) = \beta_0 \exp\big[-(\lambda_{21}lx_1 + \lambda_{12}l^{-1}x_2 - (\lambda_{21}^2l^2 + \lambda_{12}^2l^{-2})t)\big]$$

where

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \qquad \beta_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

with $c_i \in \mathbb{C}^\times$ and $l = c_1/c_2$. We have taken the linear functionals β_i to be of exponential type, although the general expression will be any linear superposition of exponentials.

We denote by $E(x_1, x_2, t)$ the function

$$E(x_1, x_2, t) := \exp\big[(\lambda_{12}k + \lambda_{21}l)x_1 + (\lambda_{21}k^{-1} + \lambda_{12}l^{-1})x_2 + (\lambda_{12}^2(k^2 - l^{-2}) + \lambda_{21}^2(k^{-2} - l^2))t\big]$$

in terms of which

$$\varphi(x_1, x_2, t) = C + \frac{A}{E(x_1, x_2, t)}$$

where $C \in \mathbb{C}$ is an arbitrary constant and the amplitude $A \in \mathbb{C}$ satisfies

$$-A = \frac{e_1c_1}{\lambda_{12}k + \lambda_{21}l} = \frac{e_2c_2}{\lambda_{21}k^{-1} + \lambda_{12}l^{-1}}.$$

The solution associated with the two-component KP equations is

$$p_{12}(x_1, x_2, t) = \lambda_{12} + \frac{e_1c_2}{A + CE(x_1, x_2, t)}$$

$$p_{21}(x_1, x_2, t) = \lambda_{21} + \frac{e_2c_1}{A + CE(x_1, x_2, t)}$$

$$U(x_1, x_2, t) = -\lambda_{12}\lambda_{21}x_1x_2 + \ln(C + A/E(x_1, x_2, t))$$

and as wavefunctions we can choose $\tilde{h} = 1/(C + AE)b$ or $h = 1/(C + AE)\beta$.

As a second illustration, we consider the case $V = \mathbb{C}^2$. If e_1 and e_2 are independent, there exists a basis where the general expression for the matrices A_1 and A_2 is

$$A_1 = \begin{pmatrix} q_1 & -\lambda_{12}k^2 \\ \lambda_{21} & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & \lambda_{12} \\ -\lambda_{21}k^{-2} & q_2 \end{pmatrix}$$

where $q_1, q_2 \in \mathbb{C}, k \in \mathbb{C}^\times$ satisfy $q_1 = k^2 q_2$. For the vectors e_1 and e_2 we have

$$e_1 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ a_2 \end{pmatrix}$$

with $a_1, a_2 \in \mathbb{C}^\times$ such that $k^2 = -a_1/a_2$. For the computation of ψ_0 it is useful to determine the eigenvalues of the matrix

$$M(x_1, x_2, t) := A_1 x_1 + A_2 x_2 + (A_1^2 + A_2^2)t.$$

The eigenvalues $\lambda_\pm$ are

$$\lambda_\pm = \tfrac{1}{2}\left(\operatorname{Tr} M \pm \sqrt{(\operatorname{Tr} M)^2 - 4 \det M}\right)$$

a formula that can be written as

$$\lambda_\pm = \zeta_\pm q_1 x_1(t) + \zeta_\mp q_2 x_2(t) - \zeta_+ \zeta_- (q_1^2 - q_2^2)t$$

where we denote

$$x_i(t) := x_i + q_i t$$

and

$$\zeta := \sqrt{\frac{1}{4} - \frac{\lambda_{12}\lambda_{21}}{q_1 q_2}}$$

$$\zeta_\pm := \tfrac{1}{2} \pm \zeta.$$

Once the eigenvalues are known it is easy to find $\psi_0 = \exp M$, recalling that

$$\exp M = \frac{M - \lambda_-}{\lambda_+ - \lambda_-} \mathrm{e}^{\lambda_+} - \frac{M - \lambda_+}{\lambda_+ - \lambda_-} \mathrm{e}^{\lambda_-}.$$

The final expression is

$$\psi_0 = \frac{1}{2\zeta} \begin{pmatrix} \zeta_+ \mathrm{e}^{\lambda_+} - \zeta_- \mathrm{e}^{\lambda_-} & -\frac{\lambda_{12}}{q_2}(\mathrm{e}^{\lambda_+} - \mathrm{e}^{\lambda_-}) \\ \frac{\lambda_{21}}{q_1}(\mathrm{e}^{\lambda_+} - \mathrm{e}^{\lambda_-}) & -\zeta_- \mathrm{e}^{\lambda_+} + \zeta_+ \mathrm{e}^{\lambda_-} \end{pmatrix}$$

and we get the formulae

$$b_1 = \frac{a_1}{2\zeta} \begin{pmatrix} \zeta_+ \mathrm{e}^{-\lambda_+} - \zeta_- \mathrm{e}^{-\lambda_-} \\ \frac{\lambda_{21}}{q_1}(\mathrm{e}^{-\lambda_+} - \mathrm{e}^{-\lambda_-}) \end{pmatrix}$$

$$b_2 = \frac{a_2}{2\zeta} \begin{pmatrix} -\frac{\lambda_{12}}{q_2}(\mathrm{e}^{-\lambda_+} - \mathrm{e}^{-\lambda_-}) \\ -\zeta_- \mathrm{e}^{-\lambda_+} + \zeta_+ \mathrm{e}^{-\lambda_-} \end{pmatrix}.$$

For the σ's we can consider, in particular, exponential expressions as those just written. Thus, let $r_1, r_2, l, c_1, c_2 \in \mathbb{C}^\times$ be such that $r_1/r_2 = l$ and $c_1/c_2 = l$. As before, we introduce

$$\xi := \sqrt{\frac{1}{4} - \frac{\lambda_{12}\lambda_{21}}{r_1 r_2}}$$

$$\xi_\pm := \tfrac{1}{2} \pm \xi$$

$$\mu_\pm := \xi_\pm r_1 (x_1 - r_1 t) + \xi_\mp r_2 (x_2 - r_2 t) + \xi_+ \xi_- (r_1^2 + r_2^2)t$$

in terms of which we write

$$\beta_1 = \frac{c_1}{2\xi}\,(\,\xi_+ \mathrm{e}^{-\mu_+} - \xi_- \mathrm{e}^{-\mu_-}, \quad \tfrac{\lambda_{12}}{r_1}(\mathrm{e}^{-\mu_+} - \mathrm{e}^{-\mu_-})\,)\,H$$

$$\beta_2 = \frac{c_2}{2\xi}\,(\,-\tfrac{\lambda_{21}}{r_2}(\mathrm{e}^{-\mu_+} - \mathrm{e}^{-\mu_-}), \quad -\xi_- \mathrm{e}^{-\mu_+} + \xi_+ \mathrm{e}^{-\mu_-}\,)\,H$$

where $H \in \mathrm{GL}(V)$ is an invertible linear operator.

Formulae for φ and the solutions to the two-component KP equations readily follow from the preceding computations. In the next section a detailed account of this particular example will be given for the DSI reduction.

These two examples illustrate the method of construction in very simple cases. However, using theorem 1 and proposition 3 we are able to construct large families of solutions of the two-component KP equations. In the rest of this section we shall concentrate on two particular examples where the space V is chosen to be infinite-dimensional, in which case equations (3.3) and (3.4) will suffice to describe the solution space.

3.1. The Klein–Gordon equation

We shall deal here with a non-zero mass Klein–Gordon equation. We first introduce the objects $s(x_1, x_2, t) \in \mathbb{C}^2 \otimes W$, $\sigma(x_1, x_2, t) \in \mathbb{C}^2 \otimes W^*$ and $\Phi(x_1, x_2, t) \in \mathrm{GL}(W)$, $\Phi_{ij}(x_1, x_2, t) \in \mathrm{L}(W)$ in terms of which our solution is constructed.

(i) Let W be a complex linear space; we shall denote by s_1 and s_2 two vector functions such that the spinor $s = (s_1, s_2)^t$ solves

$$(\partial_\gamma + m_\gamma)s = 0$$

$$(\partial_t + \Delta)s = 0$$

with $\det m_\gamma = \lambda_{12}\lambda_{21} \neq 0$.

(ii) We define linear functionals $\sigma_i(x_1, x_2, t) \in W^*$, $i = 1, 2$ such that $\sigma = (\sigma_1, \sigma_2)^t$ is a solution of

$$(\partial_\gamma + \tilde{m}_\gamma)\sigma = 0$$

$$(\partial_t - \Delta)\sigma = 0.$$

(iii) Let $\Phi(x_1, x_2, t)$ be a solution of

$$\partial_i \Phi = s_i \otimes \sigma_i \qquad i = 1, 2$$

$$\partial_t \Phi = \sum_{i=1,2} [s_i \otimes \partial_i \sigma_i - \partial_i s_i \otimes \sigma_i]$$

and we define

$$\Phi_{ij} := \Phi + s_j \otimes (\sigma_i - \varsigma_j \Phi).$$

with $\varsigma_i(x_1, x_2, t) \in W^*$ such that $\langle \varsigma_i, s_i \rangle = 1$.

According to the preceding definitions we present a scheme for the construction of solutions to the two-component KP equations in the case $\lambda_{12}\lambda_{21} \neq 0$.

Theorem 2. Let s, σ, Φ and Φ_{ij} be given as above. Then the formulae

$$p_{12} = \lambda_{12} + \langle \sigma_1, \Phi^{-1} s_2 \rangle = \lambda_{12} + \frac{\det \Phi_{12}}{\det \Phi}$$

$$p_{21} = \lambda_{21} + \langle \sigma_2, \Phi^{-1} s_1 \rangle = \lambda_{21} + \frac{\det \Phi_{21}}{\det \Phi}$$

$$U = \ln \det \Phi - \lambda_{12}\lambda_{21} x_1 x_2$$

provide a solution of (2.20)–(2.22). Moreover, the functions

$$\tilde{h}_i = \Phi^{-1} s_i \qquad i = 1, 2$$

$$h_i = \sigma_i \Phi^{-1} \qquad i = 1, 2$$

satisfy the corresponding (2.23) and (2.24) determining their components and hence their associated wavefunctions.

Proof. We choose the vector space V as the space of bi-infinite sequences in a complex linear space W, thus $V = \ell_{\mathbb{Z}}(W)$ is the set of vectors

$$\ell_{\mathbb{Z}}(W) := \big\{ \{w_n\}_{n \in \mathbb{Z}} : w_n \in W \big\}.$$

The shift operator Λ acts on V according to the formula

$$\Lambda \{w_n\}_{n \in \mathbb{Z}} = \{w_{n+1}\}_{n \in \mathbb{Z}}$$

which is an invertible operator with inverse Λ^{-1} given by

$$\Lambda^{-1} \{w_n\}_{n \in \mathbb{Z}} = \{w_{n-1}\}_{n \in \mathbb{Z}}.$$

Take $A_1 = \lambda_{12}\Lambda$, $A_2 = \lambda_{21}\Lambda^{-1}$, $e_1 = e$ and $e_2 = \Lambda^{-1} e$, where $e \neq 0$ is some constant vector in V.

From the definition of the b's we obtain the recurrence relations

$$\partial_1 b_{i,n} + \lambda_{12} b_{i,n+1} = 0$$

$$\partial_2 b_{i,n} + \lambda_{21} b_{i,n-1} = 0$$

so that, if $\lambda_{12}\lambda_{21} \neq 0$, we can write

$$b_i = \{\cdots, \underset{\substack{\downarrow \\ n=-2}}{\lambda_{21}^{-2}\partial_2^2 s_i}, \underset{\substack{\downarrow \\ n=-1}}{-\lambda_{21}^{-1}\partial_2 s_i}, \underset{\substack{\downarrow \\ n=0}}{s_i}, \underset{\substack{\downarrow \\ n=1}}{-\lambda_{12}^{-1}\partial_1 s_i}, \underset{\substack{\downarrow \\ n=2}}{\lambda_{12}^{-2}\partial_1^2 s_i}, \cdots\}$$

where $s_i := b_{i,0}$. As b satisfies (3.3) and (3.4) so do all its terms; this happens for $s = (s_1, s_2)^t$ in particular.

For the β's we choose

$$\beta_i = \{\cdots, 0, \underset{\substack{\downarrow \\ n=0}}{\sigma_i}, 0, \cdots\}$$

with σ_i functionals in W^* subject to the same equations as the β_i.

These choices give the results stated in the theorem. □

Observe that in this theorem it is necessary that $\lambda_{12}\lambda_{21} \neq 0$; in the next subsection we shall give a construction where this is not needed.

3.2. The Dirac equation

One can write more explicit expressions for the solutions when restrictions on the function ψ are imposed. In this direction, we shall analyse what these solutions are when ψ satisfies a Dirac equation and consequently the Klein–Gordon equation (2.5). We first give some basic results regarding the representation of the Clifford algebra associated with the metric introduced in (3.1).

The Clifford algebra associated with the metric $\mathrm{d}s^2 = g_{ij}\mathrm{d}x_i \otimes \mathrm{d}x_j$, defined in (3.1), generated by γ_1, γ_2 through the anticommutation relations

$$\{\gamma_i, \gamma_j\} = 2g_{ij}$$

can be represented in a linear space V by operators Γ_1, Γ_2. We introduce this notation in order to distinguish between these gamma matrices and those corresponding to the representation in $\mathbb{C}^2$. It can be easily shown in that case that the space V decomposes as a direct sum

$$V = V_1 \oplus V_2$$

where $V_1 \cong V_2 = \hat{V}$. The associated resolution of the identity is given by

$$\mathrm{id} = P_1 + P_2$$

with the projections P_i, $\left(P_i\right)^2 = P_i$ and $P_1P_2 = P_2P_1 = 0$, defined as

$$P_1 := \Gamma_1\Gamma_2, \qquad P_2 := \Gamma_2\Gamma_1.$$

Moreover, since $V_i = \Gamma_i V$ it follows that every representation is of the form

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where the matrix form is referred to the linear splitting of V into isomorphic subspaces V_1 and V_2.

Given $\lambda_{12}, \lambda_{21} \in \mathbb{C}$ we define the operator $\tilde{m}_\Gamma \in \mathrm{L}(V)$

$$\tilde{m}_\Gamma := \lambda_{21}P_1 + \lambda_{12}P_2.$$

The Dirac operator associated with this particular representation of the Clifford algebra reads

$$\partial_\Gamma := \Gamma_1\partial_1 + \Gamma_2\partial_2$$

and the Dirac equation

$$(\partial_\Gamma - \tilde{m}_\Gamma)\psi = 0 \tag{3.8}$$

for the automorphism $\psi(x_1, x_2, t) \in \mathrm{GL}(V)$ implies the Klein–Gordon equation

$$(\partial_1\partial_2 - \lambda_{12}\lambda_{21})\psi = 0.$$

The operator $Q = \lambda_{12}\lambda_{21}$ is now proportional to the identity and equation (3.8) can be written in terms of the right derivatives F_1, F_2 of ψ as

$$\Gamma_1(F_1 - \Gamma_2\lambda_{21}) + \Gamma_2(F_2 - \Gamma_1\lambda_{12}) = 0. \tag{3.9}$$

To treat the constraints (2.10) in the same fashion, we first observe that the operators A_1, A_2 in addition to (2.11) must be consistent with (3.9). One easily deduces that A_1, A_2 satisfy the constraints given by (2.11) and (3.9) if they have the form

$$A_1 = \lambda_{21}\Gamma_2 + q_1 P_1 - \lambda_{12}k^2\Gamma_1$$

$$A_2 = \lambda_{12}\Gamma_1 + q_2 P_2 - \lambda_{21}k^{-2}\Gamma_2$$

where $q_i \in \mathrm{L}(\hat{V})$, $i = 1, 2$, and $k^2 \in \mathrm{GL}(\hat{V})$, a linear invertible operator, satisfy

$$q_1 - k^2 q_2 = 0$$

and the vectors e_1, e_2 are connected by

$$e_1 - k^2\Gamma_1 e_2 = 0.$$

One can see that with this choice for the A_i, e_i, $i = 1, 2$, (3.8) holds by virtue of the constraints (2.10), which can be solved according to proposition 1.

Note that in addition to equation (3.3) the vector functions b_i are also solution of

$$(\partial_\Gamma + \tilde{m}_\Gamma)b_i = 0 \qquad i = 1, 2 \tag{3.10}$$

that follows from the particular form of the operators A_1 and A_2 when dealing with the Dirac equation for ψ. However, equations (3.3) and (3.10) imply, with respect to the linear splitting $V = V_1 \oplus V_2$, that the b's can be written as follows:

$$b_1 = \hat{b}_1 + \lambda_{21}B \tag{3.11}$$

$$b_2 = \lambda_{12}B + \hat{b}_2. \tag{3.12}$$

and this makes the Dirac equations (3.3) and (3.10) equivalent conditions. That $e_i \in V_i$, together with the definition of the b_i's, has as consequence the initial value condition

$$B(0) = 0. \tag{3.13}$$

The conditions that we have been considering so far do not completely characterize the b_i's which are of exponential type by definition. As before, a particularly interesting solution arises in an infinite-dimensional linear space V. In that case the vectors b_i are replaced by vector functions characterized uniquely by the equations written above.

In the present case theorem 2 can be formulated as follows. Let Γ_1, Γ_2 be a representation defined by the metric (3.1) in the complex linear space W and $W = \hat{W} \oplus \hat{W}$ the associated decomposition. Let s_1 and s_2 be two vector functions of x_1, x_2, t with values in W such that with respect to the linear splitting $W = \hat{W} \oplus \hat{W}$ can be written as

$$s_1 = \hat{s}_1 + \lambda_{21}S \tag{3.14}$$

$$s_2 = \lambda_{12}S + \hat{s}_2 \tag{3.15}$$

with S such that

$$S(0) = 0.$$

Theorem 3. Let $s, \sigma, \Phi, \Phi_{ij}$ denote the elements appearing in theorem 2 with s and W as above where the λ_{ij} are arbitrary numbers. Then the formulae of theorem 2 remain true.

Proof. We take

$$\hat{V} = \ell_{\mathbb{N}}(\hat{W}) := \big\{\{w_n\}_{n\geqslant 0} : w_n \in \hat{W}\big\}$$

as the space of sequences in $\hat{W}$. Choose $k^2 = 1$ and $q_1 = \Lambda$, Λ being the shift operator in $\ell_{\mathbb{N}}(\hat{W})$, so that $q_2 = \Lambda$.

With this choice we study the vectors b_1 and b_2. Due to equations (3.11) and (3.12), with respect to the splitting $V = \ell_{\mathbb{N}}(\hat{W}) \oplus \ell_{\mathbb{N}}(\hat{W})$, we can write

$$b_1 = \big\{\hat{b}_{1,n}\big\}_{n\geqslant 0} + \lambda_{21}\big\{B_n\big\}_{n\geqslant 0}$$

$$b_2 = \lambda_{12}\big\{B_n\big\}_{n\geqslant 0} + \big\{\hat{b}_{2,n}\big\}_{n\geqslant 0}.$$

The definition of $b_i = \psi_0^{-1} e_i$ gives the recurrence relations

$$\hat{b}_{1,n+1} = -\partial_1 \hat{b}_{1,n} + \lambda_{12}\lambda_{21} B_n$$

$$\hat{b}_{2,n+1} = -\partial_2 \hat{b}_{2,n} + \lambda_{12}\lambda_{21} B_n$$

$$\lambda_{12} B_{n+1} = \lambda_{12}(-\partial_1 B_n + \hat{b}_{2,n})$$

$$\lambda_{21} B_{n+1} = \lambda_{21}(-\partial_2 B_n + \hat{b}_{1,n})$$

together with equations (3.3) and (3.4) for $\hat{b}_{1,0} + B_0$ and $B_0 + \hat{b}_{2,0}$. The condition $B_0(0) = 0$ follows from $B(0) = 0$. Define $\hat{s}_i := \hat{b}_{i,0}$ and $S := B_0$. From the definition of b_i one also deduces the relations

$$\partial_1 \hat{b}_{2,n} + \lambda_{12}\lambda_{21} B_n = 0$$

$$\partial_2 \hat{b}_{1,n} + \lambda_{12}\lambda_{21} B_n = 0$$

$$\lambda_{21}(\partial_1 B_n + \hat{b}_{1,n}) = 0$$

$$\lambda_{12}(\partial_2 B_n + \hat{b}_{2,n}) = 0.$$

Observe that when $\lambda_{12}\lambda_{21} \neq 0$ we can write a symmetric recurrence relation for the B's:

$$B_{n+1} = \hat{b}_{1,n} + \hat{b}_{2,n}$$

but this equality fails when $\lambda_{12}\lambda_{21} = 0$. In this case, if one of the λ's is not zero, we could use for B the remaining non-trivial recurrence relation. If both of the λ are zero, $B = 0$.

For the β's choose

$$\beta_i = \big\{\sigma_i, 0, 0, \ldots\big\}$$

where $\sigma_i(x_1, x_2, t)$ is a linear functional in W^* such that σ is a solution to the Dirac equation and $(\partial_t - \Delta)\sigma = 0$.

Now, recalling equations (3.5) and (3.6), a proper choice of the integration constant allows us to replace the infinite-dimensional φ by Φ in order to get the desired result.

Taking into account (2.26) and (2.27), the form of the wavefunctions follows in a similar manner. $\square$

Note that when $\lambda_{12} = \lambda_{21} = 0$ the theorem above reduces to [13, theorem 3]. If this is the case $S = 0$ and s_i depends on x_i, t and takes values in W_i. The Dirac equation disappears and only the deformation equations remains for σ_i as well. Therefore, theorem 3 can be understood as a *massive* deformation of the *zero-mass* solutions given in [13]. But this algebraic *mass* should not be confused with the energy or mass of the solutions to the two-component KP equations.

Observe that the Dirac equation for s, in spinor components, reads

$$\partial_2 \hat{s}_1 + \lambda_{12}\lambda_{21} S = 0 \tag{3.16}$$

$$\partial_1 \hat{s}_2 + \lambda_{12}\lambda_{21} S = 0 \tag{3.17}$$

$$\lambda_{21}(\partial_1 S + \hat{s}_1) = 0 \tag{3.18}$$

$$\lambda_{12}(\partial_2 S + \hat{s}_2) = 0. \tag{3.19}$$

If $\lambda_{12}\lambda_{21} \neq 0$ then $\hat{s}_1$ and $\hat{s}_2$ derive from the potential $-S$, where S is a solution of

$$(\partial_1\partial_2 - \lambda_{12}\lambda_{21})S = 0. \tag{3.20}$$

Thus s is parametrized functionally in terms of an arbitrary solution S of the Klein–Gordon equation (3.20) with $S(0) = 0$. If $\lambda_{12}\lambda_{21} = 0$ then $\hat{s}_1$ is x_2-independent and $\hat{s}_2$ does not depend on x_1. Moreover, if for example, $\lambda_{12} \neq 0$, then $S = S_1(x_1, t) + S_2(x_2, t)$ and $\hat{s}_2(x_2, t) = \partial_2 S_2(x_2, t)$ defined by (3.19) in terms of S.

The reduction to the nonlinear Schrödinger system. As an illustration of our construction we shall examine briefly what the solutions are for the reduction to the nonlinear Schrödinger (NLS) equation.

Given an arbitrary vector field $X = a_1\partial_1 + a_2\partial_2$ one can ask whether $Xp_{12} = Xp_{21} = 0$ consistently with the time evolution, so that we have a one-dimensional reduction of the two-component system. In our scheme this means that $X\alpha_i = 0$, $i = 1, 2$. Taking into account the rank-one constraints for ψ, this is equivalent to

$$X\psi = \psi K$$

or

$$X\varphi + (a_1 A_1 + a_2 A_2)\varphi = \varphi K$$

for some $K \in \mathrm{L}(V)$. The operator K is determined by the initial conditions

$$K = \varphi_0^{-1}(a_1 b_{1,0} \otimes \beta_{1,0} + a_2 b_{2,0} \otimes \beta_{2,0}) + \varphi_0^{-1}(a_1 A_1 + a_2 A_2)\varphi_0$$

and the β's are now subject to

$$X\beta_i = \beta_i K.$$

One can show that there exist independent variables Y and T, the former being a linear function of x_1 and x_2 and the latter a linear function of t, and dependent variables $P_{12} := \exp(\Lambda t)p_{12}$, $P_{21} := \exp(-\Lambda t)p_{21}$, where Λ is an appropriate constant, such that the two-component KP equations reduce to the well known AKNS–ZS system or nonlinear Schrödinger (NLS) system:

$$\partial_T P_{12} - \partial_Y^2 P_{12} - 2P_{12}^2 P_{21} = 0$$

$$\partial_T P_{21} + \partial_Y^2 P_{21} + 2P_{21}^2 P_{12} = 0.$$

4. The Davey–Stewartson equations

We now slightly modify our deformation to the free Schrödinger equation

$$\mathrm{i}\partial_t \psi = (\partial_1^2 - \partial_2^2)\psi$$

so that the two-component KP equations read

$$\partial_1\partial_2 U + p_{12}p_{21} = 0$$

$$\mathrm{i}\partial_t p_{12} - \Delta p_{12} - 2p_{12}\Delta U = 0$$

$$\mathrm{i}\partial_t p_{21} + \Delta p_{21} + 2p_{21}\Delta U = 0.$$

Now, theorem 3 holds if β_i is a solution of

$$(\mathrm{i}\partial_t - \partial_1^2 + \partial_2^2)\beta_i = 0 \qquad i = 1, 2$$

and φ satisfies the modified deformation equation

$$\mathrm{i}\partial_t\varphi = b_1 \otimes \partial_1\beta_1 - \partial_1 b_1 \otimes \beta_1 - b_2 \otimes \partial_2\beta_2 + \partial_2 b_2 \otimes \beta_2.$$

Theorems 2 and 3 require that s_i and σ_i, $i = 1, 2$ be solutions of

$$(\mathrm{i}\partial_t + \partial_1^2 - \partial_2^2)s_i = 0$$

$$(\mathrm{i}\partial_t - \partial_1^2 + \partial_2^2)\sigma_i = 0$$

for $i = 1, 2$, and also that Φ satisfies the deformation

$$\mathrm{i}\partial_t\Phi = s_1 \otimes \partial_1\sigma_1 - \partial_1 s_1 \otimes \sigma_1 - s_2 \otimes \partial_2\sigma_2 + \partial_2 s_2 \otimes \sigma_2.$$

4.1. The Davey–Stewartson I equations

The DSI reduction appears when $x_1 = \xi, x_2 = \eta \in \mathbb{R}$ and

$$p_{12} = \varepsilon\bar{p}_{21} =: p \quad \varepsilon = \pm 1 \qquad \nabla U(\xi, \eta, t) \in \mathbb{R}^2$$

(the bar denotes complex conjugate) which implies the differential equations

$$\partial_\xi\partial_\eta U + \varepsilon|p|^2 = 0 \tag{4.1}$$

$$\mathrm{i}\partial_t p - \Delta p - 2p\Delta U = 0. \tag{4.2}$$

These equations are just the DSI in its defocusing, $\varepsilon = 1$, and focusing, $\varepsilon = -1$, cases.

The problem to tackle here is which data A_i, β_i are suitable for this reduction. If the complex linear space V is furnished with a scalar product and $^\dagger : V \to V^*$ denotes the standard antilinear isomorphism generated by this scalar product, a possible solution to this question is as follows.

Proposition 4. If

$$\lambda := \lambda_{12} = \varepsilon\bar{\lambda}_{21}$$

$$\beta_1 = b_1^\dagger H$$

$$\beta_2 = \varepsilon b_2^\dagger H$$

$$\varphi_0^\dagger H = H\varphi_0$$

where $H = H^\dagger$ is a Hermitian operator and φ_0 denotes the value of $\varphi(\xi, \eta, t)$ at $\xi = \xi_0, \eta = \eta_0, t = t_0$, then the functions p_{12}, p_{21} and U satisfy

$$p_{12} = \varepsilon \bar{p}_{21} \qquad \nabla U(\xi, \eta, t) \in \mathbb{R}^2.$$

Proof. Observe that $\beta = (\beta_1, \beta_2)^t$ is a solution of $(\partial_\gamma + \tilde{m}_\gamma)\beta = 0$ and $(\mathrm{i}\partial_t - \partial_\xi^2 + \partial_\eta^2)\beta = 0$ because $b = (b_1, b_2)^t$ is a solution of $(\partial_\gamma + m_\gamma)b = 0$ and $(\mathrm{i}\partial_t + \partial_\xi^2 - \partial_\eta^2)b = 0$.

From definition 2 we have

$$p_{12} = \lambda_{12} + \langle b_1^\dagger, H\varphi^{-1} b_2\rangle$$

$$p_{21} = \lambda_{21} + \varepsilon\langle b_2^\dagger, H\varphi^{-1} b_1\rangle$$

so that

$$\varepsilon \bar{p}_{21} = \lambda_{12} + \left\langle b_1^\dagger, \left(\varphi^\dagger\right)^{-1} H b_2\right\rangle. \tag{4.3}$$

From the differential equations defining φ, which read

$$\partial_\xi \varphi = b_1 \otimes b_1^\dagger H$$

$$\partial_\eta \varphi = \varepsilon b_2 \otimes b_2^\dagger H$$

$$\mathrm{i}\partial_t \varphi = \left[b_1 \otimes \partial_\xi b_1^\dagger - \partial_\xi b_1 \otimes b_1^\dagger - \varepsilon(b_2 \otimes \partial_\eta b_2^\dagger - \partial_\eta b_2 \otimes b_2^\dagger)\right] H$$

one can see that

$$\partial_\xi\left(\varphi^\dagger H - H\varphi\right) = 0 \tag{4.4}$$

$$\partial_\eta\left(\varphi^\dagger H - H\varphi\right) = 0 \tag{4.5}$$

$$\partial_t\left(\varphi^\dagger H - H\varphi\right) = 0. \tag{4.6}$$

Therefore, if the initial condition φ_0 is such that

$$\varphi_0^\dagger H - H\varphi_0 = 0$$

then this condition can be extended, by means of equations (4.4)–(4.6) for every ξ, η, t, so that

$$\varphi^\dagger H - H\varphi = 0$$

Then equation (4.3) implies $\varepsilon \bar{p}_{21} = p_{12}$.

A similar argument ensures that $p_{ii} = \bar{p}_{ii}$, so that ∇U takes values in $\mathbb{R}^2$. □

Let us consider the reduction to DSI of the simple example considered previously with $V = \mathbb{C}$. The result is that if we introduce the function

$$E(\xi, \eta, t) := \exp\left(2\lambda \cos\alpha \left[k\xi(t) + \varepsilon k^{-1}\eta(t)\right]\right)$$

where

$$\xi(t) := \xi + 2\lambda k \sin\alpha\, t$$

$$\eta(t) := \eta - 2\varepsilon\lambda k^{-1} \sin\alpha\, t$$

then

$$p(\xi, \eta, t) = \lambda + \frac{kBH}{CE(\xi, \eta, t) - kB/(\lambda(\mathrm{e}^{\alpha} + \varepsilon \mathrm{e}^{-\alpha}))}$$

where $k, B > 0$ are positive numbers, $C, H, \lambda \in \mathbb{R}$ are real numbers and $\alpha \in [0, 2\pi)$, is the amplitude for a solution of the DSI equations which happens to be a one line soliton solution of DSI [21].

The reduction, in the defocusing case, of the $V = \mathbb{C}^2$ example follows. Given $q_1, q_2 < 0$ we define

$$Q := \sqrt{\frac{1}{4} - \frac{|\lambda|^2}{q_1 q_2}}$$

and we choose q_1, q_2 such that $0 \leqslant Q \leqslant \frac{1}{2}$, that is $q_1 q_2 \geqslant 4|\lambda|^2$. It proves to be convenient to introduce $Q_\pm := \frac{1}{2} \pm Q$. We also define $\xi(t) := \xi - \mathrm{i} q_1 t$ and $\eta(t) := \eta + \mathrm{i} q_2 t$, and the corresponding eigenvalues reads $\lambda_\pm := Q_\pm q_1 \xi(t) + Q_\mp q_2 \eta(t) + \mathrm{i} Q_+ Q_- (q_1^2 - q_2^2) t$. The vectors b_1 and b_2 in terms of which the solution is constructed are

$$b_1 = \frac{q_1}{2Q} \begin{pmatrix} Q_+ \exp(-\lambda_+) - Q_- \exp(-\lambda_-) \\ \frac{\bar{\lambda}}{q_1} (\exp(-\lambda_+) - \exp(-\lambda_-)) \end{pmatrix}$$

$$b_2 = \frac{q_2}{2Q} \begin{pmatrix} \frac{\lambda}{q_2} (\exp(-\lambda_+) - \exp(-\lambda_-)) \\ Q_- \exp(-\lambda_+) - Q_+ \exp(-\lambda_-) \end{pmatrix}.$$

The associated β's that give rise to the DSI reduction are found by using the prescription $\beta_i = b_i^\dagger H$, for $i = 1, 2$, with $H^\dagger = H$.

The fundamental matrix is $\varphi = \mathrm{id} + \phi H$, where $\varphi_0 = \mathrm{id}$, now $(\xi_0, \eta_0) = -\infty$ so that b_i goes to zero at that initial point. Here the Hermitian matrix $\phi = (\phi_{ij})$ is defined by

$$\phi_{11} := -\frac{q_1}{8Q^2} \exp(-(q_1\xi + q_2\eta))[Q_+ \exp(-2Q(q_1\xi - q_2\eta)) + Q_- \exp(2Q(q_1\xi - q_2\eta)) \\ -4Q_+Q_- \cos(2Q(q_1^2 + q_2^2)t)]$$

$$\phi_{22} := -\frac{q_2}{8Q^2} \exp(-(q_1\xi + q_2\eta))[Q_- \exp(-2Q(q_1\xi - q_2\eta)) + Q_+ \exp(2Q(q_1\xi - q_2\eta)) \\ -4Q_+Q_- \cos(2Q(q_1^2 + q_2^2)t)]$$

$$\phi_{12} := -\frac{\lambda}{8Q^2} \exp(-(q_1\xi + q_2\eta))[\exp(-2Q(q_1\xi - q_2\eta)) + \exp(2Q(q_1\xi - q_2\eta)) \\ -2(\cos(2Q(q_1^2 + q_2^2)t) + 2\mathrm{i}Q \sin((2Q(q_1^2 + q_2^2)t))].$$

The principal tau function is $\det\varphi$, which in our case reads

$$\det\varphi = 1 + \mathrm{Tr}(\phi H) + \det H \det\phi.$$

One can check that $\det\phi = q_1 q_2 \exp(-2(q_1\xi + q_2\eta))/4$, and if $H = (H_{ij})$ we can write

$$\det\varphi = 1 - \frac{1}{8Q^2} \exp(-(q_1\xi + q_2\eta))\big[\Sigma_+ \exp(-2Q(q_1\xi - q_2\eta)) \\ \Sigma_- \exp(2Q(q_1\xi - q_2\eta)) - \Sigma_0(t)\big] + \frac{q_1 q_2 \det H}{4} \exp(-2(q_1\xi + q_2\eta))$$

where

$$\Sigma_+ := H_{11}q_1Q_+ + H_{22}q_2Q_- + 2\lambda\,\mathrm{Re}(H_{21})$$

$$\Sigma_- := H_{11}q_1Q_- + H_{22}q_2Q_+ + 2\lambda\,\mathrm{Re}(H_{21})$$

$$\Sigma_0(t) := 4\Big[(H_{11}q_1 + H_{22}q_2)Q_+Q_-\cos(2Q(q_1^2+q_2^2)t)$$
$$+\lambda\,\mathrm{Re}\left\{H_{21}(\cos(2Q(q_1^2+q_2^2)t) + 2\mathrm{i}Q\sin(2Q(q_1^2+q_2^2)t))\right\}\Big].$$

This expression allows us to find the potential U through the formula $U = -|\lambda|^2\xi\eta + \ln\det\varphi$. Moreover, the modulus of the amplitude $|p|$ is determined by $|p|^2 = |\lambda|^2 - \partial_\xi\partial_\eta\ln\det\varphi$. When $\lambda = 0$ one recovers the general formulae for the one-dromion solution first found in [3], an exponential localized solution in all directions. As $|\lambda|^2$ goes from 0 to $q_1q_2/4$ one finds a one-complex-parameter deformation of the one-dromion solution of DSI. Motivated by the mass character of the parameter λ in the Klein–Gordon and Dirac equations in which it appears, we call them *massive deformations*. Although the solution does not move in the plane a general constant velocity can be assigned by performing a Galilean transformation. The resulting velocity is connected with the imaginary part of the new q's.

The term Σ_0 gives a time dependence, so that $|p|$ has a pulsation with period given by $T = \pi/(Q(q_1^2+q_2^2))$. This breather character for $|p|$ disappears when $\Sigma_0 = 0$, for example in the one-dromion solution.

From the form of $\det\varphi$, where four independent exponentials appear generically, one readily concludes that this solution cannot be a two line-soliton solution of the DSI equation [21]. However, the representation with *Mathematica* of $|p|$ strongly suggests a nonlinear superposition of a one-dromion with a two-line soliton, with the dromion living in the cross of the line solitons. The solution remains bounded, but now the asymptotic values at infinity are non-zero, depending these asymptotics on the value of λ. The two-line soliton disappears when $\lambda = 0$ and the dromion almost disappears when the mass λ is large enough. Moreover, *Mathematica* shows that only the angle of the two lines defined by ΔU, which coincide with those defined by the two-line soliton, depend asymptotically on λ.

A more detailed study of this solution and its generalizations to the N-massive dromion solutions seems to be of interest besides the study of the standard line solitons. The N-dromion solution [14] appears when $V = \mathbb{C}^{2N}$ and $\lambda = 0$, so massive deformations appears by allowing λ to be distinct from 0.

Proposition 4 allows us to apply theorems 2 and 3 to the DSI equation. As before we introduce the functions $s(\xi,\eta,t) \in \mathbb{C}^2 \otimes W$, $\Phi(\xi,\eta,t) \in \mathrm{GL}(W)$ and $\tilde{\Phi}(\xi,\eta,t)$ in terms of which our solutions are constructed.

(i) Let W be a complex linear space and define two vector functions s_1, s_2 solutions of

$$(\partial_\gamma + m_\gamma)s = 0$$

$$(\mathrm{i}\partial_t + \partial_\xi^2 - \partial_\eta^2)s = 0$$

with $s = (s_1, s_2)^t$ and $m_\gamma = \mathrm{diag}(\lambda, \varepsilon\bar{\lambda})$. We shall distinguish between two different types of data:

(a) Type I. In this case $\lambda \neq 0$.

(b) Type II. In this case $\lambda \in \mathbb{C}$ can be chosen equal to zero and W must be chosen of the form $W = \hat{W} \oplus \hat{W}$, where $\hat{W}$ is a complex linear space, and the vector functions

s_1 and s_2 decompose with respect to the linear splitting $W = \hat{W} \oplus \hat{W}$ as

$$s_1 = \hat{s}_1 + \varepsilon\bar{\lambda}S$$

$$s_2 = \lambda S + \hat{s}_2$$

with S satisfying

$$S(0) = 0.$$

(ii) Let $\Phi(\xi, \eta, t)$ be a solution of

$$\partial_\xi \Phi = s_1 \otimes s_1^\dagger H$$

$$\partial_\eta \Phi = \varepsilon s_2 \otimes s_2^\dagger H$$

$$\mathrm{i}\partial_t \Phi = \left[s_1 \otimes \partial_\xi s_1^\dagger - \partial_\xi s_1 \otimes s_1^\dagger - \varepsilon(s_2 \otimes \partial_\eta s_2^\dagger - \partial_\eta s_2 \otimes s_2^\dagger)\right]H$$

where H is a Hermitian operator, for which the initial condition Φ_0 satisfies

$$\Phi_0^\dagger H - H\Phi_0 = 0$$

we define

$$\tilde{\Phi} := \Phi + s_2 \otimes (s_1^\dagger H - \varsigma\Phi).$$

with $\varsigma(\xi, \eta, t) \in W^*$ such that $\langle \varsigma, s_2 \rangle = 1$.

Our construction of solutions can be stated in terms of these functions.

Theorem 4. Let s, Φ and $\tilde{\Phi}$ be as just described, with s either of type I or type II, then

$$p = \lambda + \langle s_1^\dagger, H\Phi^{-1}s_2 \rangle = \lambda + \frac{\det \tilde{\Phi}}{\det \Phi}$$

$$U = \ln\det\Phi - \varepsilon|\lambda|^2\xi\eta$$

are a solution of the DSI equations given by (4.1) and (4.2). The vector functions

$$\tilde{h}_i = \Phi^{-1}s_i$$

satisfy the corresponding equations (2.23) and (2.24) and their components give the associated wavefunctions.

Observe that for the adjoint wavefunction we have $h_i = \tilde{h}_i^\dagger H$, which follows from the particular form of the σ's and the relation $H\Phi^{-1} = \left(\Phi^\dagger\right)^{-1}H$. We also remark that, because the global gauge invariance, λ can be chosen to be real without loss of generality.

The type II case is a *massive* extension of [13, theorem 5] where the case $\lambda = 0$ was considered. Recall that the *zero-mass* case [13, theorem 5] was first discovered in [7] by spectral means following the inverse spectral analysis of [5] for DSI; it also appears in [11], where it was independently derived by direct methods. The case $\lambda = 0$ contains the well known dromion [3, 14] and gausson [7] solutions of DSI.

4.2. The DSII *reduction*

For the DSII case we take $\bar{x}_1 = x_2$, $\nabla U = (U_x, U_y) \in \mathbb{R}^2$ where $x_1 = z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, and $\varepsilon \bar{p}_{21} = p_{12} = p$. The equations are now

$$\Delta U + 2\varepsilon |p|^2 = 0 \tag{4.7}$$

$$\tfrac{1}{2}\mathrm{i}\partial_t p - (\partial_x^2 - \partial_y^2)p - 2p(\partial_x^2 - \partial_y^2)U = 0. \tag{4.8}$$

Equations (4.1) and (4.8) constitute the DSII in its defocusing ($\varepsilon = 1$) and focusing ($\varepsilon = -1$) cases. Here we use the notation $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$.

Proposition 5. If

$$\bar{e}_2 = P e_1$$

$$\bar{\lambda}_{21} = \varepsilon \lambda_{12} =: \varepsilon \lambda$$

$$\bar{A}_2 = P A_1 P^{-1}$$

$$\bar{\beta}_1 = \beta_2 T$$

where $P, T \in \{A \in \mathrm{GL}(V) : \bar{A}A = \varepsilon\}$, and the initial value φ_0 of φ satisfies

$$\bar{\varphi}_0 - \varepsilon P \varphi_0 T = 0$$

then

$$p_{12} = \varepsilon \bar{p}_{21}$$

$$\nabla U(x, y, t) \in \mathbb{R}^2.$$

Proof. This choice of A_i and e_i is consistent with conditions (2.12) and compatible with (2.11). One can check that $\bar{\psi}_0 = P\psi_0 P^{-1}$, thus

$$\bar{b}_2 = P b_1 \tag{4.9}$$

$$\bar{b}_1 = \varepsilon P b_2 \tag{4.10}$$

and for the β's we have

$$\bar{\beta}_2 = \varepsilon \beta_1 T \tag{4.11}$$

$$\bar{\beta}_1 = \beta_2 T. \tag{4.12}$$

The equations defining φ are

$$\partial\varphi = b_1 \otimes \beta_1 \tag{4.13}$$

$$\bar{\partial}\varphi = b_2 \otimes \beta_2 \tag{4.14}$$

$$\mathrm{i}\partial_t \varphi = b_1 \otimes \partial\beta_1 - \partial b_1 \otimes \beta_1 - b_2 \otimes \bar{\partial}\beta_2 + \bar{\partial} b_2 \otimes \beta_2 \tag{4.15}$$

and the complex conjugate equations, once equations (4.9) and (4.10) are used, give the relations

$$\bar{\partial}(\bar{\varphi} - \varepsilon P \varphi T) = 0 \tag{4.16}$$

$$\partial(\bar{\varphi} - \varepsilon P \varphi T) = 0 \tag{4.17}$$

$$\partial_t(\bar{\varphi} - \varepsilon P \varphi T) = 0. \tag{4.18}$$

Thus, the initial condition

$$\bar{\varphi}_0 - \varepsilon P \varphi_0 T = 0$$

can be extended up to $\bar{\varphi} = \varepsilon P \varphi T$.

Observe that this condition implies $\det \bar{\varphi} = \pm \det P \det T \det \varphi$. But, $|\det P| = |\det T| = 1$. Therefore, there exists a constant θ such that $\det \varphi(z, \bar{z}, t) \in \exp(\mathrm{i}\theta)\mathbb{R}$, so that ∇U takes real values. Hence

$$\varepsilon \bar{p}_{21} = \varepsilon \langle \bar{\beta}_2, \bar{\varphi}^{-1} \bar{b}_1 \rangle = \varepsilon \langle \varepsilon \beta_1 T, \varepsilon T^{-1} \varphi^{-1} P^{-1} \varepsilon P b_1 \rangle = p_{12}.$$

□

This proposition allows us to reduce theorems 2 and 3 to the DSII equation. As done previously, we introduce the elements $s(z, \bar{z}, t) \in \mathbb{C}^2 \otimes W$, $\sigma(z, \bar{z}, t) \in \mathbb{C}^2 \otimes W^*$, $\Phi(z, \bar{z}, t) \in \mathrm{GL}(W)$ and $\tilde{\Phi}(z, \bar{z}, t) \in \mathrm{L}(W)$ in terms of which the solutions are constructed. Again we will distinguish between different types of possibilities.

(i) We introduce some linear algebra:
  - (a) Type I. $P, T \in \{A \in \mathrm{GL}(W) : \bar{A}A = \varepsilon\}$ are linear operators over the complex linear space W.
  - (b) Type II. As in type I, but the operator P must be chosen in an appropriate manner:
  $$P = \Gamma_1 + \varepsilon \Gamma_2$$
  where we are considering a representation of the Clifford algebra given by the metric (3.1) over the complex linear space $W = \hat{W} \oplus \hat{W}$.

(ii) Take $s(z, \bar{z}, t)$ a W-valued solution of

$$\bar{\partial} s + \varepsilon \bar{\lambda} \bar{P} \bar{s} = 0$$

$$(\mathrm{i}\partial_t + \partial^2 - \bar{\partial}^2) s = 0$$

  - (a) Type I. $\lambda \neq 0$.
  - (b) Type II. We can take $\lambda = 0$, but $s(z, \bar{z}, t)$ is a vector function that with respect to the splitting $W = \hat{W} \oplus \hat{W}$ can be written as $s(z, \bar{z}, t) = \hat{s}(z, \bar{z}, t) + \varepsilon \bar{\lambda} S(z, \bar{z}, t)$, with $\varepsilon \bar{S} = S$, satisfying $S(0) = 0$.

(iii) Choose $\sigma(z, \bar{z}, t) \in W^*$ a solution of

$$\bar{\partial} \sigma + \varepsilon \lambda \bar{\sigma} \bar{T} = 0$$

$$(\mathrm{i}\partial_t - \partial^2 + \bar{\partial}^2) \sigma = 0.$$

(iv) Let $\Phi(z, \bar{z}, t) \in \mathrm{GL}(W)$ be a solution of

$$\partial \Phi = s \otimes \sigma$$

$$\bar{\partial} \Phi = \varepsilon \bar{P} \bar{s} \otimes \bar{\sigma} \bar{T}$$

$$\mathrm{i}\partial_t \Phi = s \otimes \partial \sigma - \partial s \otimes \sigma - \varepsilon \bar{P} \big[\bar{s} \otimes \bar{\partial} \bar{\sigma} - \bar{\partial} \bar{s} \otimes \bar{\sigma} \big] \bar{T}.$$

such that the initial condition Φ_0 satisfies

$$\Phi_0 - \varepsilon P \Phi_0 T = 0$$

and define

$$\tilde{\Phi} := \Phi + \bar{P}\bar{s} \otimes (\sigma - \varsigma\Phi)$$

with $\varsigma(z, \bar{z}, t) \in W^*$ such that $\langle \varsigma, \bar{P}\bar{s} \rangle = 1$.

We remark that in type II the function Φ has the form

$$\Phi := \phi \oplus \bar{\phi}\bar{T}$$

with $\phi(z, \bar{z}, t) \in \mathrm{L}(\hat{W} \oplus \hat{W}, \hat{W})$ solution of

$$\partial\phi = \hat{s} \otimes \sigma$$

$$\bar{\partial}\phi = \varepsilon\lambda S \otimes \bar{\sigma}\bar{T}$$

$$\mathrm{i}\partial_t\phi = \hat{s} \otimes \partial\sigma - \partial\hat{s} \otimes \sigma - \varepsilon\lambda\big[S \otimes \bar{\partial}\bar{\sigma} - \bar{\partial}S \otimes \bar{\sigma}\big]\bar{T}.$$

We can now give the solutions of DSII according to our scheme.

Theorem 5. Let s, σ, Φ and $\tilde{\Phi}$ be as above. Then the functions

$$U = \ln\det\Phi - \varepsilon|\lambda|^2(x^2 + y^2)$$

$$p = \lambda + \langle \sigma, \Phi^{-1}s \rangle = \lambda + \frac{\det\tilde{\Phi}}{\det\Phi}$$

solve the DSII equations given by (4.7) and (4.8). In terms of the functions $\tilde{h} := \Phi^{-1}s$, $h := \sigma\Phi^{-1}$ one can construct

$$\tilde{h}_1 := h \quad \tilde{h}_2 := T\bar{\tilde{h}} \qquad \text{and} \qquad h_1 = h \quad h_2 = \varepsilon\bar{h}P$$

that satisfy equations (2.23) and (2.24), their components giving the wavefunctions.

Proof. *Type I.* From equations (4.10), (4.9) (3.3) we deduce $\bar{\partial}s + \varepsilon\bar{\lambda}\bar{P}\bar{s} = 0$. The Dirac equation for $(\sigma_1, \sigma_2)^t$ can be written, once we recall equations (4.11) and (4.12), as the differential equation $\bar{\partial}\sigma + \varepsilon\lambda\bar{\sigma}\bar{T} = 0$. The wavefunctions have the form

$$\tilde{h}_1 := \Phi^{-1}s$$

$$\tilde{h}_2 := \Phi^{-1}\bar{P}\bar{s}$$

but

$$\Phi^{-1} = T\bar{\Phi}^{-1}\bar{P}^{-1}$$

from where our expression follows. The remaining one follows in a similar fashion.
Type II. From equations (4.10), (4.9) and the particular form of P it follows that $\varepsilon\bar{S} = S$. Use the notation $\hat{s}_1 = \hat{s}$ and $\sigma = \sigma_1$. The Dirac equation for $(\sigma_1, \sigma_2)^t$ is equivalent, once we recall equations (4.11) and (4.12), to the differential equation $\bar{\partial}\sigma + \varepsilon\lambda\bar{\sigma}\bar{T} = 0$. □

The construction given in theorem 5 contains, in type I, the one-line soliton given in [1]. For type II theorem 5 is a *massive* extension of the *zero-mass* case given in [13, theorem 7]. The deformations determined by this *mass* on the solutions contained in [13, theorem 7], which contains the soliton solutions of [2] in particular, will be analysed elsewhere.

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